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# An analysis of a family of rational maps containing integrable and non-integrable difference analogue of the logistic equation

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**Abstract.** The iteration of rational maps of degree two is discussed for functions which arise from a difference method approximating the logistic equation and we interpolate between integrable and nonintegrable maps. We obtain a necessary and sufficient condition for the statement that the Julia set is contained in the set of real numbers and a sufficient condition for the Julia set to be a Cantor set. The information dimensions of the Julia sets are calculated numerically to show how they change as the parameter of the interpolation approaches the integrable limit.

## 1. Introduction

In the study of nonlinear phenomena the transition between chaotic and non-chaotic behaviour is one of the most difficult and also important subjects to be clarified in order to understand their dynamics. We have proposed [1] a simple model of the type of logistic equation which interpolates integrable and non-integrable maps and enables us to study the transition analytically. Since the Julia set of a map characterizes non-integrability and does not exist in an integrable map, we are interested in knowing how it behaves as a parameter, which interpolates between the two phases, approaches the critical value.

The iteration of rational functions $f_{\mu,\gamma}(x)$ given by

$$f_{\mu,\gamma}(x) = \frac{\mu x(1-\gamma x)}{1+\mu(1-\gamma)x} \qquad \mu > 1 \text{ and } 0 \leqslant \gamma \leqslant 1 \tag{1}$$

is our original concern in this paper. The functions $f_{\mu,\gamma}(x)$ appear in a difference method approximating the logistic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} u = au(1-u) \tag{2}$$

in two different methods, one of which preserves integrability of the differential equation while the other violates integrability under the discretization.

The well known map which generates chaos is derived from (2) by considering the time variable $t$ as being discrete. Let $\tau$ be the unit time interval and consider the following equation due to the Euler difference method:

$$\frac{u_{n+1}-u_n}{\tau} = au_n(1-u_n). \tag{3}$$

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Let $x_n$ and $\mu$ be given by

$$x_n = \frac{a\tau}{1+a\tau}u_n \qquad \mu = 1 + a\tau \tag{4}$$

then (3) is written in the form

$$x_{n+1} = \mu x_n(1 - x_n). \tag{5}$$

An iteration of this map is well known [2, 3] to behave chaotically when $\mu$ is greater than 3.615 47. . . .

The difference method of (2) is not unique but there exist infinitely many ways. Among other candidates, we are interested in a method which preserves integrability which is possessed by the original equation (2). Such a method has been studied by Morisita 20 years ago [4, 5]. In his method (2) was replaced by

$$x_{n+1} = \mu x_n(1 - x_{n+1}). \tag{6}$$

It is not difficult to convince ourselves that this equation reduces to the differential equation (2) in the continuous time limit. The integrability of this map is apparent if we solve (6) for $x_{n+1}$:

$$x_{n+1} = \mu \frac{x_n}{1+\mu x_n}. \tag{7}$$

Since this is a Möbius map, the iteration of this map preserves its form of the map. In fact the solution is given by

$$x_n = \mu^n \frac{x_0}{1 + \mu(1-\mu^n/1-\mu)x_0} \tag{8}$$

for an arbitrary initial value $x_0$. The solution of this form coincides with the solution of (2) in the continuous limit.

In our previous paper [1] we have shown that the integrable map (6) can be obtained by a reduction from the Hirota bilinear difference equation. The Hirota bilinear difference equation itself is satisfied by every solution of the KP hierarchy of soliton equations. Therefore the integrability of equation (6) has the same origin with the soliton equations.

Now there arises a question of what is the source which causes such a big difference of the behaviour of solutions to (5) and (6), both of which are obtained by discretizations of the same equation (2). We want to understand their behaviour from a common background. We investigate these particular maps not because they are special, but because we are interested in them because one is a prototype of non-integrable systems and the other of completely integrable systems.

It should be emphasized here that the complete integrability of a nonlinear map, of which we are concerned in this paper, differs from a non-chaotic behaviour. For instance, a solution of the so-called logistic map (5) persues a periodic motion and is stable if the value of $\mu$ is less than 3.615 47. . . , but the system is not integrable in the sense that we cannot solve the map analytically. On the other hand, a completely integrable map is a system in which all of the solutions can be obtained analytically. We have a variety of completely integrable nonlinear systems known as soliton systems. Although they play central roles in physics and other fields, they occupy only a small part of nonlinear phenomena and are surrounded by chaos. The transition between completely integrable and non-integrable behaviour must be distinguished from the transition between stable behaviour and chaotic behaviour in one non-integrable system. We consider in this paper the former transition.

From a phenomenological point of view it is quite difficult to judge if a stable motion is caused by completely integrable dynamics or non-integrable dynamics. Therefore, we ask the question of how can one characterize a completely integrable system among other nonlinear systems. Instead of answering this question phenomenologically, we would like to propose here that we consider the dynamical system in the complex plane and study a Julia set of the iteration of the map.

A Julia set of the logistic map (5) appears on the real axis when $\mu$ is greater than 3.615 47... and the chaotic behaviour of the map is observed; otherwise, it remains off the real axis in the complex plane even if a solution is periodic and stable. A Julia set does not exist if a map is completely integrable. Therefore the existence of a Julia set is a way of discriminating between a completely integrable system and a non-integrable system.

In order to clarify the transition between an integrable map and a non-integrable map it would be useful to study a system which interpolates them, rather than studying them in parallel. For this purpose we combine the maps (5) and (6) into

$$x_{n+1} = \mu x_n\{1 - \gamma x_n - (1-\gamma)x_{n+1}\} \tag{9}$$

that is,

$$x_{n+1} = f_{\mu,\gamma}(x_n) \tag{10}$$

where $f_{\mu,\gamma}$ is given by (1).

In the case in which $\gamma = 1$, $f_{\mu,1}(x)$ is the logistic map (5). In the other extreme case $\gamma = 0$, the sequence of iterated images $\{f_{\mu,0}^{\circ n}(x)\}_{n\geqslant 0}$ converges to $1 - 1/\mu$ as $n$ tends to infinity, for any $\mu > 1$ and $x > 0$, and we can say that 'chaos' does not occur. Our class of rational functions $f_{\mu,\gamma}$ includes both the extreme cases. We note that the Schwartzian derivative of the map $f_{\mu,\gamma}$ is negative at $x \neq -1/(\mu(1-\gamma))$ for each $\mu > 1$ and $\gamma \in [0, 1]$, and that the map $f_{\mu,\gamma}$ is unimodal in the interval $[0, 1/\gamma]$. Under an additional condition, the map $f_{\mu,\gamma}$ is a $S$-unimodal map on the interval $[0, 1/\gamma]$ [6].

The functions $f_{\mu,\gamma}(x)$ given by (1) will be extended to the functions $f_{\mu,\gamma}(z)$ with a complex variable $z$:

$$f_{\mu,\gamma}(z) = \frac{\mu z(1-\gamma z)}{1+\mu(1-\gamma)z}. \tag{11}$$

The rational functions $f_{\mu,\gamma}(z)$ can be regarded as holomorphic maps from the Riemann sphere $\hat{C}$ to $\hat{C}$.

Since we are interested in clarifying the transition between integrable and non-integrable maps, the main purpose of this paper is to study the iteration of $f_{\mu,\gamma}$ with small $\gamma$. It is, however, more convenient to consider, instead of the $f_{\mu,\gamma}$, the map $F_{\mu,\gamma}$ defined by

$$F_{\mu,\gamma}(z) = \varphi^{-1} \circ f_{\mu,\gamma} \circ \varphi(z) \tag{12}$$

where $\varphi(z)$ is the Möbius map given by

$$\varphi(z) = z + 1 - \frac{1}{\mu}.$$

Note that $\varphi(0) = 1 - 1/\mu$. The origin is a fixed point of $F_{\mu,\gamma}$, since $1 - 1/\mu$ is a fixed point of $f_{\mu,\gamma}$. The origin will be shown in section 2 to be an attracting fixed point of $F_{\mu,\gamma}$ for $\gamma < \frac{1}{2}$.

We are mainly concerned with the Julia set $J(F_{\mu,\gamma})$ and see their features at small $\gamma$. In section 3, a necessary and sufficient condition will be obtained to ensure that the Julia set $J(F_{\mu,\gamma})$ is included in the set of real numbers, and the immediate basin of the origin will turn out to be equal to $\hat{C} - J(F_{\mu,\gamma})$ for small $\gamma$. The Julia set $J(F_{\mu,\gamma})$ will be shown, in section 4, to be a Cantor set for small $\gamma$. In this case, $F_{\mu,\gamma}$ is a map called type E in

[7]. The boundary of the immediate basin of the origin of our map must be classified as the fractal case according to the classification of basin boundaries in [8].

The existence of a Julia set on the real axis means that physically we observe chaos when the parameters are properly fixed. It is then desirable to know how they look, particularly when $\gamma$ is small. To supply information of their behaviour we present numerical calculations of the information dimension and the box dimension of the Julia set for some values of the parameters. It will be shown that the Julia set becomes dilute as the parameter $\gamma$ approaches zero, which is the critical value of the transition.

## 2. Preliminaries

The family of rational functions, which will be discussed in this paper, is given by

$$F_{\mu,\gamma}(z) = z\frac{-\mu\gamma z + 1 + \gamma - \gamma\mu}{(1-\gamma)\mu z + \mu + \gamma - \gamma\mu} \tag{13}$$

with parameters $\mu > 1$ and $\gamma \in [0, 1]$. $F_{\mu,\gamma}$ is a holomorphic map from the Riemann sphere $\hat{C}$ to itself, which has degree two for $\gamma > 0$. In the case in which $\gamma = 0$, $F_{\mu,\gamma}(z)$ is a Möbius transformation (bijective rational map).

The fixed points of the map $F_{\mu,\gamma}$ for $\gamma > 0$ are 0, $\infty$ and $z_0 = -1 + 1/\mu$, and $F_{\mu,0}$ has the two fixed points 0 and $z_0$. In order to make clear the stability and instability of the fixed points, we need the following lemma, which is obtained by an elementary calculation.

*Lemma 2.1.*

$$F'_{\mu,\gamma}(0) = \frac{1 + \gamma - \gamma\mu}{\mu - \gamma\mu + \gamma}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{1}{F_{\mu,\gamma}(1/z)}\right\}\Big|_{z=0} = -\frac{1-\gamma}{\gamma} \qquad \text{for } \gamma > 0$$

and

$$F'_{\mu,\gamma}(z_0) = \mu. \tag{14}$$

Lemma 2.1 implies the following.

The fixed points 0 and $\infty$ are attracting and repelling, respectively, if $0 < \gamma < \frac{1}{2}$, attracting and neutral if $\gamma = \frac{1}{2}$, attracting and attracting if $\mu(2\gamma - 1) < 2\gamma + 1$ and $\gamma > \frac{1}{2}$, neutral and attracting if $\mu(2\gamma - 1) = 2\gamma + 1$, and repelling and attracting if $\mu(2\gamma - 1) > 2\gamma + 1$. The fixed point $z_0$ is always repelling.

Let $G_1(z)$ and $G_2(z)$ be holomorphic maps from $\hat{C}$ to $\hat{C}$. If $\varphi^{-1} \circ G_1 \circ \varphi = G_2$ holds for some Möbius transformation $\varphi$, then $G_1$ and $G_2$ are called analytically conjugate, and the dynamical systems $\{G_1^{\circ n} : n \geqslant 0\}$ and $\{G_2^{\circ n} : n \geqslant 0\}$ turn out to be holomorphically the same.

Next, we will give the condition for two rational functions in our family (13) to be analytically conjugate.

*Lemma 2.2.* Distinct maps $F_{\mu,\gamma}(z)$ and $F_{\mu',\gamma'}(z)$ are analytically conjugate if and only if the both conditions

$$\mu = \mu' \tag{15}$$

and

$$\frac{\gamma + \gamma'}{2} = \frac{1}{2}\left(1 + \frac{1}{\mu - 1}\right) \tag{16}$$

hold.

*Proof.* Suppose that there exists a Möbius transformation $\varphi$ such that the equality

$$\varphi^{-1} \circ F_{\mu,\gamma} \circ \varphi = F_{\mu',\gamma'} \tag{17}$$

holds for suitable pairs $(\mu, \gamma)$ and $(\mu', \gamma')$.

Then, $\varphi$ maps the fixed points of $F_{\mu',\gamma'}$ to that of $F_{\mu,\gamma}$, and it keeps the multipliers at the fixed points.

Put $z_0' = \varphi^{-1}(z_0)$. The above fact implies that $F'_{\mu',\gamma'}(z_0') = \mu > 1$. $z_0'$ cannot be equal to 0 or $\infty$ because $F'_{\mu',\gamma'}(0) < 1$ and the multiplier of $F'_{\mu',\gamma'}$ at $\infty$ is negative as are known from lemma 2.1. That is, $F'_{\mu',\gamma'}(z_0') = \mu'$. Hence equality (15) holds and it follows $z_0 = z_0'$ from $z_0' = -1 + 1/\mu'$.

Thus, there exists at most one non-identical Möbius transformation $\varphi$ satisfying (17), and it is written in the form

$$\varphi(z) = \frac{(-1 + 1/\mu)^2}{z}. \tag{18}$$

An elementary calculation gives us the fact that the transformation $\varphi$ given by (18) satisfies (17) for $(\mu, \gamma)$ and $(\mu, \gamma')$ if and only if the equality (16) holds. □

## 3. The Julia set $J(F_{\mu,\gamma})$ for small $\gamma$

Let $G(z)$ be any holomorphic map from the Riemann sphere $\hat{C}$ to $\hat{C}$. That is, $G(z)$ is a rational function on $\hat{C}$. First, we will give the definition of the Julia set and the Fatou set of $G$.

*Definition.* Let $z_0 \in \hat{C}$ be fixed. If there exists some neighbourhood $U$ of $z_0$ so that sequence of iterates $\{G^{\circ n}\}$ restricted to $U$ forms a normal family, then we say that $z_0$ belongs to the Fatou set of $G$. Otherwise, if no such neighbourhood exists, we say that $z_0$ belongs to the Julia set $J(G)$.

Next, we recall the basic statement on the Julia set.

Suppose that $z_0' \in J(G)$. Then, the Julia set $J(G)$ is equal to the closure of the set of all iterated preimages

$$\{z : G^{\circ n}(z) = z_0' \text{ for some } n \geqslant 0\}.$$

This statement will play an important role in the proof of theorem 3.1.

Now, we are in a position to state one of our main results.

*Theorem 3.1.* Suppose that $\gamma < \frac{1}{2}$. The Julia set $J(F_{\mu,\gamma})$ is contained in the real numbers $\mathbb{R}$ if and only if

$$\mu \geqslant \left(\frac{2\gamma}{1 - 2\gamma}\right)^2.$$

*Proof.* Define $\tilde{F}_{\mu,\gamma}(z)$ by

$$\tilde{F}_{\mu,\gamma}(z) = \varphi^{-1} \circ F_{\mu,\gamma}(z) \circ \varphi$$

where $\varphi(z) = \frac{1}{2}$. We have

$$\tilde{F}_{\mu,\gamma}(z) = z\,\frac{(\mu + \gamma - \gamma\mu)z + (1 - \gamma)\mu}{(1 + \gamma - \gamma\mu)z - \mu\gamma}. \tag{19}$$

Note that $J(F_{\mu,\gamma}) \subset \mathbb{R}$ is equivalent to $J(\tilde{F}_{\mu,\gamma}) \subset \mathbb{R}$.

We can see that $1/z_0 = -\mu/\mu - 1 \in J(\tilde{F}_{\mu,\gamma})$, since $\varphi(z_0) = 1/z_0$ is a repelling fixed point of $\tilde{F}_{\mu,\gamma}$. Consequently, our problem is equivalent to finding a condition to ensure that the set of all preimages of $1/z_0$

$$A = \{z : \tilde{F}^{\circ n}_{\mu,\gamma}(z) = 1/z_0 \text{ for some } n \geqslant 0\} \tag{20}$$

is included in the real numbers $\mathbb{R}$.

It is easy to see that $\tilde{F}^{-1}_{\mu,\gamma}(1/z_0)$ equals the set $\{1/z_0, z_1\}$, where $z_1 = 1/(1/\gamma + 1/\mu - 1)$. For simplicity, we put $z_2 = \gamma\mu/(1 + \gamma - \gamma\mu)$. Note that

$$\tilde{F}_{\mu,\gamma}(z_2) = \infty \in \hat{C}.$$

The intervals $(-\infty, z_2)$ and $(z_2, +\infty)$ in $\mathbb{R}$ are denoted by $E_i$ $(i = 1, 2)$ in the following way:

$$E_1 = (-\infty, z_2) \text{ and } E_2 = (z_2 + \infty) \text{ if } z_2 > 0$$

$$E_1 = (z_2 + \infty) \text{ and } E_2 = (-\infty, z_2) \text{ if } z_2 < 0.$$

The case $z_2 = 0$ is excluded, because $z_2 = 0$ means $\gamma = 0$.

By a standard calculus, we see that $\max_{x\in E_1} \tilde{F}_{\mu,\gamma}(x)$ and $\min_{x\in E_2} \tilde{F}_{\mu,\gamma}(x)$ exist, and that

$$\max_{x\in E_1} \tilde{F}_{\mu,\gamma}(x) = \frac{1}{\mu/\left(\sqrt{1+\Gamma\mu}+1\right)^2 (1/\gamma) - 1 + (1/\mu)} \tag{21}$$

and

$$\min_{x\in E_2} \tilde{F}_{\mu,\gamma}(x) = \frac{1}{\mu/\left(\sqrt{1+\Gamma\mu}+1\right)^2 (1/\gamma) - 1 + (1/\mu)} \tag{22}$$

where $\Gamma = (1 - \gamma)/\gamma$.

Note that $\tilde{F}_{\mu,\gamma}(x)$ is unimodal in the interval $[1/z_0, z_1] \subset E_1$, that

$$\min_{x\in[1/z_0,z_1]} \tilde{F}_{\mu,\gamma}(x) = \tilde{F}_{\mu,\gamma}\left(\frac{1}{z_0}\right) = \tilde{F}_{\mu,\gamma}(z_1) = \frac{1}{z_0}$$

and that the condition $\gamma < \frac{1}{2}$ implies

$$\min_{x\in E_2} \tilde{F}_{\mu,\gamma}(x) > z_1. \tag{23}$$

Suppose that $\max_{x\in E_1} \tilde{F}_{\mu,\gamma}(x) < z_1$. Then, the equality

$$\tilde{F}_{\mu,\gamma}(x) = z_1$$

does not have real solutions, that is, $\tilde{F}^{-1}_{\mu,\gamma}(z_1)$ are not real.

Consequently, the condition

$$\max_{x\in E_1} \tilde{F}_{\mu,\gamma}(x) \geqslant z_1 \tag{24}$$

is necessary for $A \subset \mathbb{R}$, where the set $A$ is given by (20).

Suppose that (24) holds. Then, for any real number $a$ in the interval $[1/z_0, z_1]$, both the solutions of the equality

$$\tilde{F}_{\mu,\gamma}(x) = a$$

belong to the interval $[1/z_0, z_1]$. Hence, we see that the set $A$ of all preimages of $1/z_0$ is included in $\mathbb{R}$. Thus, we obtain that the condition (24) is sufficient for $A \subset \mathbb{R}$.

By (21) and (24), we see that the condition

$$\mu \geqslant \left(\frac{2\gamma}{1 - 2\gamma}\right)^2$$

is necessary and sufficient for $J(F_{\mu,\gamma}) \subset \mathbb{R}$. Thus, the proof is complete. □

Making use of lemma 2.2, we can find the next statement for the case $\gamma > \frac{1}{2}$.

*Corollary 3.2.* Suppose that $\gamma > \frac{1}{2}$. The Julia set $J(F_{\mu,\gamma})$ is included in $\mathbb{R}$ if and only if $\mu \geqslant (2\gamma/(1-2\gamma))^2$.

Now, we discuss the immediate basin of the origin for the map $F_{\mu,\gamma}$ with parameters satisfying $\gamma < \frac{1}{2}$ and $\mu \geqslant (2\gamma/(1-2\gamma))^2$. Let $\Omega$ be the immediate basin of the origin, which is the attracting fixed point of $F_{\mu,\gamma}$. The set $\Omega$ is known to be an open set in the Riemann sphere $\hat{C}$ which contains some neighbourhood of the origin. By theorem 3.1, we obtain the following theorem.

*Theorem 3.3.* Suppose that $\gamma < \frac{1}{2}$, and that $\mu \geqslant (2\gamma/(1-2\gamma))^2$. Then, the immediate basin $\Omega$ of the origin is equal to the open set $\hat{C} - J(F_{\mu,\gamma})$.

*Proof.* First, let $\Omega$ be the basin of the origin. By the definition of Julia sets, we see that the boundary $\partial\Omega$ of $\Omega$ should be included in $J(F_{\mu,\gamma})$. Suppose that $z' \in C - \mathbb{R}$ does not belong to $\Omega$. Then, it turns out that there exists $z'' \in \partial\Omega$ on the segment connecting $z'$ and the origin. Thus, $z'' \in J(F_{\mu,\gamma})$ and $z''$ is not real. This is a contradiction. Consequently, $C - \mathbb{R} \subset \Omega$.

Next, we suppose that $x \in \mathbb{R}$ does not belong to $\Omega$. Then, we see that $x \in \partial\Omega$, because any neighbourhood of $x$ contains a point of $\Omega$. Hence, $x \in J(F_{\mu,\gamma})$. Consequently, $x \in \mathbb{R}$ and $x \notin J(F_{\mu,\gamma})$ imply $x \in \Omega$. Since $\Omega$ is connected, the proof is complete. □

*Remark 3.4.* Suppose that $\gamma < \frac{1}{2}$, and that $\mu \geqslant (2\gamma/(1-2\gamma))^2$. The proof of theorem 3.1 shows $J(\tilde{F}_{\mu,\gamma}) \subset [1/z_0, z_1]$. That is, we see that $J(F_{\mu,\gamma}) \subset (-\infty, z_0] \cup [1/\gamma + 1/\mu - 1, \infty) \cup \{\infty\}$.

## 4. The Julia set as a Cantor set

In this section, we study the Julia set $J(F_{\mu,\gamma})$ in the case that $J(F_{\mu,\gamma})$ is included in the set of real numbers $\mathbb{R}$. Namely, we suppose that $\gamma \neq \frac{1}{2}$, and that $\mu \geqslant (2\gamma/(1-2\gamma))^2$. We are concerned with conditions on $\mu$ and $\gamma$ under which $J(F_{\mu,\gamma})$ is a Cantor set.

Our argument will be based on the following statement. Let $R$ be a rational map of degree $d$, where $d \geqslant 2$, and let $\zeta$ be a (super) attracting fixed point of $R$. If all of the critical points of $R$ lie in the immediate basin of $\zeta$, then $J(R)$ is a Cantor set (theorem 9.8.1 in [9]).

Making use of this statement, we obtain the next theorem.

*Theorem 4.1.* Suppose that $\gamma < \frac{1}{2}$, and that $\mu > (2\gamma/(1-2\gamma))^2$. Then, the Julia set $J(F_{\mu,\gamma})$ is a Cantor set.

*Proof.* It is sufficient to verify that all critical points of $F_{\mu,\gamma}$ lie in the immediate basin of the origin. Obviously, $F_{\mu,\gamma}$ has two distinct critical points, which are denoted by $\alpha$ and $\beta$. One of them is easily shown to lie in the interval $(-1+1/\mu, 1/\gamma - 1 + 1/\mu)$. Denote it by $\alpha$. We see that the critical point $\alpha$ belongs to the immediate basin $\Omega$ of the origin, because the interval $(-1+1/\mu, 1/\gamma - 1 + 1/\mu)$ is contained in $\Omega$. In order to show that the other critical point $\beta$ lies in the immediate basin $\Omega$, we need the notation used in the proof of theorem 3.1. Obviously, $\tilde{F}_{\mu,\gamma}(1/\beta) = \max_{x\in E_1} \tilde{F}_{\mu,\gamma}(x)$. Therefore, if $\tilde{F}_{\mu,\gamma}(1/\beta) = \max_{x\in E_1} \tilde{F}_{\mu,\gamma}(x) > z_1$, the point $1/\beta$ does not belong to $J(\tilde{F}_{\mu,\gamma})$. Hence, if $\mu > (2\gamma/(1-2\gamma))^2$, the point $\beta$ does not lie in $J(F_{\mu,\gamma})$. Thus, the proof is complete. □

Theorem 4.1 and lemma 2.2 imply the next corollary.

*Corollary 4.2.* Suppose that $\gamma > \frac{1}{2}$, and that $\mu > (2\gamma/(1-2\gamma))^2$. Then, $J(F_{\mu,\gamma})$ is a Cantor set.

*Remark.* Under the conditions $\gamma \neq \frac{1}{2}$ and $\mu > (2\gamma/(1-2\gamma))^2$, $F_{\mu,\gamma}$ is a map of type E in [7].

## 5. Conclusion

We have studied iteration of the rational map which interpolates integrable and non-integrable discretizations of the logistic equation. In particular we have shown that the Julia set is contained on the real axis when the interpolation parameter $\gamma$ is sufficiently small. This means that chaos exists physically as the map approaches from a positive value the critical point $\gamma = 0$, where the map becomes integrable. If we want to clarify further the transition between integrable and non-integrable maps near the critical point we must know more details of the nature of the Julia sets, such as the Hausdorff dimension.

In order to supply this information we will present some numerical calculations of the Julia sets in the following. We have calculated the Julia sets of the map $\tilde{F}_{\mu,\gamma}(z)$ for various values of $\gamma$ which satisfy $\mu \geqslant [2\gamma/(1-2\gamma)]^2$ when $\mu = 9$.

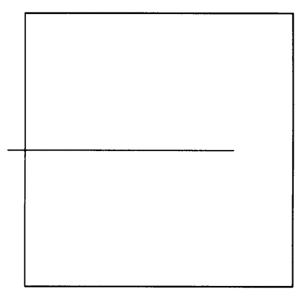

**Figure 1.** $\gamma = 0.375:\ d_{\mathrm{i}} = 0.752,\ d_{\mathrm{b}} = 0.862.$

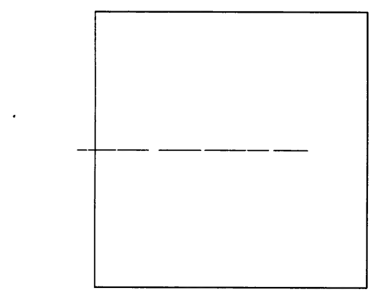

**Figure 2.** $\gamma = 0.374:\ d_{\mathrm{i}} = 0.738,\ d_{\mathrm{b}} = 0.847.$

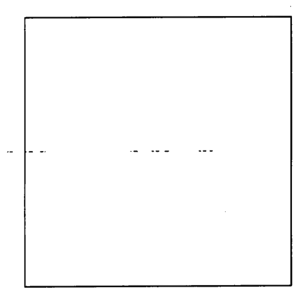

**Figure 3.** $\gamma = 0.3:\ d_{\mathrm{i}} = 0.534,\ d_{\mathrm{b}} = 0.577.$

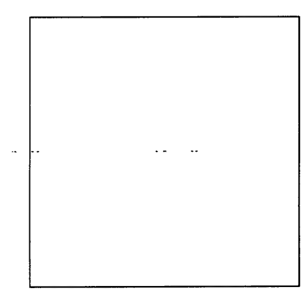

**Figure 4.** $\gamma = 0.2:\ d_{\mathrm{i}} = 0.434,\ d_{\mathrm{b}} = 0.446.$

In the figures the square box indicates the fixed scale at $z = \pm 1$ and $z = \pm \mathrm{i}$. It is clear from the figures that the Julia sets are contained in the real axis as is claimed in the text.

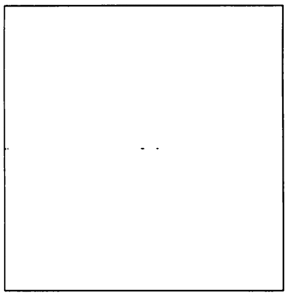

**Figure 5.** $\gamma = 0.1: \ d_\mathrm{i} = 0.347, \ d_\mathrm{b} = 0.354$.

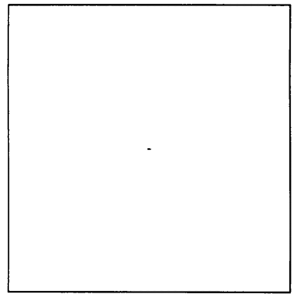

**Figure 6.** $\gamma = 0.01: \ d_\mathrm{i} = 0.277, \ d_\mathrm{b} = 0.288$.

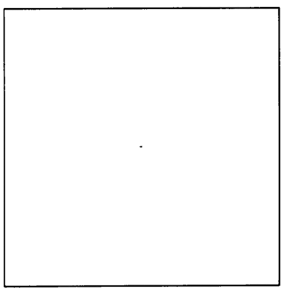

**Figure 7.** $\gamma = 0.001: \ d_\mathrm{i} = 0.245, \ d_\mathrm{b} = 0.259$.

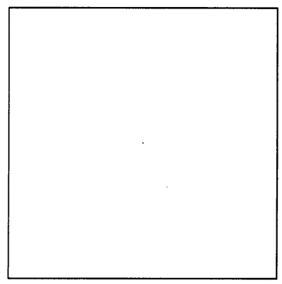

**Figure 8.** $\gamma = 10^{-10}: \ d_\mathrm{i} = 0.195, \ d_\mathrm{b} = 0.223$.

If $\gamma$ does not satisfy the condition $\mu \geqslant [2\gamma/(1-2\gamma)]^2$ by a small value or $\gamma$ becomes complex the Julia set extends outside of the real axis. Further observation reveals that the region on the real axis where the Julia set exists becomes smaller as $\gamma$ becomes smaller. In fact the inverse map of $\tilde{F}_{\mu,\gamma}(z)$ becomes

$$\frac{1}{2\mu}\left(z - \mu \pm \sqrt{(z-\mu)^2}\right) \tag{25}$$

and hence it iterates the map starting from 0 to the set

$$0, -1, -1 - \frac{1}{\mu}, -1 - \frac{1}{\mu} - \frac{1}{\mu^2}, \ldots, -\mathrm{e}^{1/\mu}.$$

This indicates that the Julia set becomes dilute as the parameter of interpolation approaches the critical point. Our numerical calculation of the information dimension $d_\mathrm{i}$ as well as the box dimension $d_\mathrm{b}$ seems to support this claim.

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